Supplement to *Interval Spacing*

Greg Kreider

Derivation of Equations

We use

$$S_1 = \frac{n!}{(i-w-1)!(w-1)!(n-i)!}$$

throughout to denote the scaling factor before the density function, as in the main text.

The density of the interval spacing is

$$f_{D_{i,w}}(y) = S_1 \int_{-\infty}^{\infty} \{F_x(x)\}^{i-w-1} \{F_x(x+y) - F_x(x)\}^{w-1} \{1 - F_x(x+y)\}^{n-i} \tag{1}$$
$$\times f_x(x) f_x(x+y) dx \tag{2}$$

The expected value and variance follow from the first and second moments.

$$E\Big\{D_{i,w}\Big\} = \int_0^{\infty} y \ f_{D_{i,w}}(y) \ dy \tag{3}$$

$$V\Big\{D_{i,w}\Big\} = E\Big\{D_{i,w}^2\Big\} - E\Big\{D_{i,w}\Big\}^2 = \int_0^{\infty} y^2 \ f_{D_{i,w}}(y) \ dy - E\Big\{D_{i,w}\Big\}^2 \tag{4}$$

Interval Spacing for Uniform Variates

The variate's density and distribution functions are

$$f_{unif}(x) = \frac{1}{b-a} \tag{U.1}$$

$$F_{unif}(x) = \frac{x-a}{b-a} \tag{U.2}$$

for $a \leq x \leq b$, with $f(x)$ zero outside this range and $F(x)$ zero below and one above.

Density Function

$$\begin{aligned}
f_{D_{i,w,unif}}(y) &= S_1 \int_{-\infty}^{\infty} \left\{\frac{x-a}{b-a}\right\}^{i-w-1} \left\{\frac{x+y-a}{b-a} - \frac{x-a}{b-a}\right\}^{w-1} \left\{1 - \frac{x+y-a}{b-a}\right\}^{n-i} f(x) \ f(x+y) \ dx \\
&= S_1 \int_{-\infty}^{\infty} \left\{\frac{x-a}{b-a}\right\}^{i-w-1} \left\{\frac{y}{b-a}\right\}^{w-1} \left\{\frac{b-a-(x+y-a)}{b-a}\right\}^{n-i} \left(\frac{1}{b-a}\right)^2 dx \\
&= S_1 y^{w-1} \left(\frac{1}{b-a}\right)^n \int_0^{b-y} \{x-a\}^{i-w-1} \{b-y-x\}^{n-i} dx
\end{aligned}$$

Using [1, (3.196.3)]

$$\int_{\alpha}^{\beta} (x-\alpha)^{\mu-1}(\beta-x)^{\nu-1} dx = (\beta-\alpha)^{\mu+\nu-1} B(\mu,\nu) = (\beta-\alpha)^{\mu+\nu-1} \ \frac{(\mu-1)!(\nu-1)!}{(\mu+\nu-1)!}$$

with $\alpha = a$, $\beta = b - y$, $\mu = i - w$, $\nu = n - i + 1$ and $\mu + \nu - 1 = n - w$,

$$f_{D_{i,w,unif}}(y) = \frac{n!}{(i-w-1)!(w-1)!(n-i)!}\ y^{w-1}\left(\frac{1}{b-a}\right)^n (b-y-a)^{n-w}\ \frac{(i-w-1)!(n-i)!}{(n-w)!}$$

$$= \frac{n!}{(w-1)!(n-w)!}\ y^{w-1}\left(\frac{1}{b-a}\right)^n (b-y-a)^{n-w} \tag{U.3}$$

If $w = 1$ this matches the spacing result [2, (4)].

Expected Interval Spacing

$$E\Big\{D_{i,w,unif}\Big\} = \int_0^\infty y\ f_{D_{i,w,unif}}(y)\ dy$$

$$= \int_0^\infty \frac{n!}{(w-1)!(n-w)!}\ y^{w}\left(\frac{1}{b-a}\right)^n (b-y-a)^{n-w}\ dy$$

$$= \frac{n!}{(w-1)!(n-w)!}\left(\frac{1}{b-a}\right)^n \int_0^{b-a} y^{w}\,(b-a-y)^{n-w}\ dy$$

Again using [1, (3.196.3)] with $\alpha = 0$, $\beta = b - a$, $\mu = w + 1$, $\nu = n - w + 1$, and $\mu + \nu - 1 = n + 1$,

$$E\Big\{D_{i,w,unif}\Big\} = \frac{n!}{(w-1)!(n-w)!}\left(\frac{1}{b-a}\right)^n (b-a)^{n+1}\ \frac{w!(n-w)!}{(n+1)!}$$

$$= w\ \frac{b-a}{n+1} \tag{U.4}$$

If $w = 1$ this reduces to the spacing result [2, (5)].

Variance of Interval Spacing

$$E\Big\{D^2_{i,w,unif}\Big\} = \int_0^\infty y^2\ f_{D_{i,w,unif}}(y)\ dy$$

$$= \int_0^\infty \frac{n!}{(w-1)!(n-w!)}\ y^{w+1}\left(\frac{1}{b-a}\right)^n (b-y-a)^{n-w}\ dy$$

$$= \frac{n!}{(w-1)!(n-w)!}\left(\frac{1}{b-a}\right)^n \int_0^{b-a} y^{w+1}\,(b-a-y)^{n-w}\ dy$$

Using [1, (3.196.3)] with $\alpha = 0$, $\beta = b - a$, $\mu = w + 2$, $\nu = n - w + 1$, and $\mu + \nu - 1 = n + 2$,

$$E\Big\{D^2_{i,w,unif}\Big\} = \frac{n!}{(w-1)!(n-w)!}\left(\frac{1}{b-a}\right)^n (b-a)^{n+2}\ \frac{(w+1)!(n-w)!}{(n+2)!}$$

$$= \frac{w(w+1)}{(n+1)(n+2)}\,(b-a)^2$$

Then

$$V\Big\{D^2_{i,w,unif}\Big\} = E\Big\{D^2_{i,w}\Big\} - E\Big\{D_{i,w}\Big\}^2$$

$$= \frac{w(w+1)}{(n+1)(n+2)}\,(b-a)^2 - \frac{w^2}{(n+1)^2}\,(b-a)^2$$

$$= \frac{(w^2+w)(n+1) - w^2(n+2)}{(n+1)^2(n+2)}\,(b-a)^2$$

$$= \frac{w(n+1) + w^2(n+1-(n+2))}{n+2}\left(\frac{b-a}{n+1}\right)^2$$

$$= \frac{w(n+1-w)}{n+2}\left(\frac{b-a}{n+1}\right)^2 \tag{U.5}$$

With $w = 1$ this is the same as the spacing variance.

Interval Spacing for Exponential Variates

The variate's density and distribution functions are

$$f_{exp}(x) = \lambda e^{-\lambda x} \tag{E.1}$$
$$F_{exp}(x) = 1 - e^{-\lambda x} \tag{E.2}$$

for $x \geq 0$.

Density Function

$$\begin{aligned}
f_{D_{i,w,exp}}(y) &= S_1 \int_{-\infty}^{\infty} \left\{1 - e^{-\lambda x}\right\}^{i-w-1} \left\{\left(1 - e^{-\lambda(x+y)}\right) - \left(1 - e^{-\lambda x}\right)\right\}^{w-1} \left\{1 - \left(1 - e^{-\lambda(x+y)}\right)\right\}^{n-i} \\
&\qquad \cdot \lambda\, e^{-\lambda x}\ \lambda\, e^{-\lambda(x+y)}\ dx \\
&= S_1 \int_{-\infty}^{\infty} \left\{1 - e^{-\lambda x}\right\}^{i-w-1} \left\{e^{-\lambda x} - e^{-\lambda(x+y)}\right\}^{w-1} \left\{e^{-\lambda(x+y)}\right\}^{n-i} \lambda^2\, e^{-\lambda x}\ e^{-\lambda(x+y)}\ dx \\
&= S_1 \int_{-\infty}^{\infty} \left\{1 - e^{-\lambda x}\right\}^{i-w-1} e^{-\lambda(w-1)x} \left\{1 - e^{-\lambda y}\right\}^{w-1}\ e^{-\lambda(n-i)x}\ e^{-\lambda(n-i)y}\ \lambda^2\ e^{-2\lambda x}\ e^{-\lambda y}\ dx \\
&= S_1\ e^{-\lambda(n-i)y}\ e^{-\lambda y} \lambda^2 \int_{-\infty}^{\infty} \left\{1 - e^{-\lambda x}\right\}^{i-w-1} e^{-\lambda(w-1)x}\ e^{-\lambda(n-i)x}\ e^{-2\lambda x}\ dx \\
&= S_1 \left\{1 - e^{-\lambda y}\right\}^{w-1} \int_{0}^{\infty} \left\{1 - e^{-\lambda x}\right\}^{i-w-1} e^{-\lambda(n-i+w+1)x} dx
\end{aligned}$$

With [1, (3.312.1)]

$$\int_0^{\infty} \left(1 - e^{-x/\beta}\right)^{\nu-1} e^{-\mu x} dx = \beta B(\beta\mu, \nu) = \beta\ \frac{(\beta\mu - 1)!(\nu - 1)!}{(\beta\mu + \nu - 1)!}$$

we have $\beta = 1/\lambda$, $\nu = i - w$, $\mu = (n - i + w + 1)\lambda$. Then $\beta\mu = n - i + w + 1$ and $\beta\mu + \nu - 1 = n$. Substituting,

$$\begin{aligned}
f_{D_{i,w,exp}}(y) &= S_1 \lambda^2 e^{-\lambda(n-i+1)y} \left\{1 - e^{-\lambda y}\right\}^{w-1} \frac{1}{\lambda} \frac{(n - i + w)!(i - w - 1)!}{n!} \\
&= \frac{n!}{(i - w - 1)!(w - 1)!(n - i)!} \frac{(n - i + w)!(i - w - 1)!}{n!}\ \lambda \left\{e^{-\lambda y}\right\}^{n-i+1} \left\{1 - e^{-\lambda y}\right\}^{w-1} \\
&= \frac{(n - i + w)!}{(w - 1)!(n - i)!}\ \lambda \left\{e^{-\lambda y}\right\}^{n-i+1} \left\{1 - e^{-\lambda y}\right\}^{w-1}
\end{aligned} \tag{E.3}$$

Note that the pre-factor is

$$w \binom{n - i + w}{n - i}$$

If $w = 1$ this simplifies to [2, (7)].

Expected Interval Spacing

$$\begin{aligned}
E\left\{D_{i,w,exp}\right\} &= \int_0^{\infty} y\ f_{D_{i,w,exp}}(y)\ dy \\
&= \int_0^{\infty} y\ \frac{(n - i + w)!}{(w - 1)!(n - i)!}\ \lambda \left\{e^{-\lambda y}\right\}^{n-i+1} \left\{1 - e^{-\lambda y}\right\}^{w-1} dy \\
&= \frac{(n - i + w)!}{(w - 1)!(n - i)!}\ \lambda \int_0^{\infty} y \left\{e^{-\lambda y}\right\}^{n-i+1} \left\{1 - e^{-\lambda y}\right\}^{w-1} dy
\end{aligned}$$

Using [1, (3.432)]

$$\int_0^{\infty} x^{\nu-1} e^{-\mu x} \left(e^{-x} - 1\right)^{\alpha} dx = \Gamma(\nu) \sum_{k=0}^{\alpha} (-1)^k \binom{\alpha}{k}\ \frac{1}{(\alpha + \mu - k)^{\nu}}$$

and making the variable change $y = x/\lambda$, $dy = dx/\lambda$ so that

$$\frac{(n-i+w)!}{(w-1)!(n-i)!}\ \lambda \int_0^\infty \frac{x}{\lambda} \left\{1-e^{-x}\right\}^{w-1} \left\{e^{-x}\right\}^{n-i+1} \frac{1}{\lambda}\ dx$$
$$= \frac{(n-i+w)!}{(w-1)!(n-i)!}\frac{1}{\lambda}(-1)^{w-1} \int_0^\infty x \left\{e^{-x}-1\right\}^{w-1} \left\{e^{-x}\right\}^{n-i+1} dx$$

we identify $\nu = 2$, $\alpha = w-1$, $\mu = n-i+1$, and $\alpha + \mu = n-i+w$, giving

$$E\Big\{D_{i,w,exp}\Big\} = \frac{(n-i+w)!}{(w-1)!(n-i)!}\frac{1}{\lambda}(-1)^{w-1}\sum_{k=0}^{w-1}(-1)^k\binom{w-1}{k}\frac{1}{(n-i+w-k)^2} \tag{E.4}$$

If $w = 1$ the sum reduces to one term and we have $1/\lambda(n-i+1)$, which is the same as [2, (8)].

Variance of Interval Spacing

$$\begin{aligned}
E\Big\{D^2_{i,w,exp}\Big\} &= \int_0^\infty y^2 f_{D_{i,w,exp}}(y)\ dy \\
&= \int_0^\infty y^2\ \frac{(n-i+w)!}{(w-1)!(n-i)!}\ \lambda \left\{1-e^{-\lambda y}\right\}^{w-1} \left\{e^{-\lambda y}\right\}^{n-i+1} dy \\
&= \frac{(n-i+w)!}{(w-1)!(n-i)!}\ \lambda \int_0^\infty \frac{1}{\lambda^2}\ x^2 \left\{1-e^{-x}\right\}^{w-1} \left\{e^{-x}\right\}^{n-i+1} \frac{1}{\lambda}\ dx \\
&= \frac{(n-i+w)!}{(w-1)!(n-i)!}\frac{1}{\lambda^2}(-1)^{w-1} \int_0^\infty x^2 \left\{e^{-x}-1\right\}^{w-1} \left\{e^{-x}\right\}^{n-i+1} dx
\end{aligned}$$

where we have again changed variables in the third line. From [1, (3.412)] with now $\nu = 3$, $\mu = n-i+1$, $\alpha = w-1$, $\mu + \alpha = n-i+w$, and $\Gamma(\nu) = 2$,

$$E\Big\{D^2_{i,w,exp}\Big\} = \frac{(n-i+w)!}{(w-1)!(n-i)!}\frac{2}{\lambda^2}\sum_{k=0}^{w-1}(-1)^k\binom{w-1}{k}\frac{1}{(n-i+w-k)^3} \tag{E.5}$$

With the series we cannot easily square this, so the variance uses (4) with (E.4) and it. If $w = 1$ the value simplifies to $2/(\lambda(n-i+1))^2$, which after subtraction reduces the numerator to 1 and matches the variance of D_i.

To evaluate this series, you must use a high precision math library.

Interval Spacing for Logistic Variates

The variate's density and distribution functions are

$$f_{logis}(x) = \frac{1}{\sigma}\frac{e^{-(x-\mu)/\sigma}}{\left(1+e^{-(x-\mu)/\sigma}\right)^2} \tag{L.1}$$
$$F_{logis}(x) = \frac{1}{1+e^{-(x-\mu)/\sigma}} \tag{L.2}$$

for all x. Substituting $z = (x-\mu)/\sigma$ and $dz = dx/\sigma$, and for the shifted functions $v = y/\sigma$

$$\begin{aligned}
f_{logis}(x) &\to \frac{e^{-z}}{\sigma\left(1+e^{-z}\right)^2} \\
F_{logis}(x) &\to \frac{1}{1+e^{-z}} \\
f_{logis}(x+y) &\to \frac{e^v e^{-z}}{\sigma\left(e^v+e^{-z}\right)^2} \\
F_{logis}(x+y) &\to \frac{e^v}{e^v+e^{-z}}
\end{aligned}$$

Density Function

$$\begin{aligned}
f_{D_{i,w,logis}}(y) &= S_1 \int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-z}}\right\}^{i-w-1} \left\{\frac{e^v}{e^v+e^{-z}} - \frac{1}{1+e^{-z}}\right\}^{w-1} \left\{1 - \frac{e^v}{e^v - e^{-z}}\right\}^{n-i} \\
&\qquad \cdot \frac{1}{\sigma} \frac{e^{-z}}{(1+e^{-z})^2} \frac{1}{\sigma} \frac{e^v e^{-z}}{(e^v+e^{-z})^2}\ \sigma\ dz \\
&= \frac{S_1}{\sigma} \int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-z}}\right\}^{i-w-1} \left\{\frac{e^v + e^v e^{-z} - e^v - e^{-z}}{(e^v+e^{-z})(1+e^{-z})}\right\}^{w-1} \left\{\frac{e^v + e^{-z} - e^v}{e^v + e^{-z}}\right\}^{n-i} \\
&\qquad \cdot e^{-z} \left\{\frac{1}{1+e^{-z}}\right\}^2 e^v e^{-z} \left\{\frac{1}{e^v+e^{-z}}\right\}^2 dz \\
&= \frac{S_1}{\sigma} \int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-z}}\right\}^{i-w-1} \left\{\frac{(e^v-1)\, e^{-z}}{(e^v+e^{-z})(1+e^{-z})}\right\}^{w-1} \left\{\frac{e^{-z}}{e^v+e^{-z}}\right\}^{2} \\
&\qquad \cdot \left\{e^{-z}\right\}^2 \left\{\frac{1}{1+e^{-z}}\right\}^2 e^v \left\{\frac{1}{e^v+e^{-z}}\right\}^2 dz \\
&= \frac{S_1}{\sigma}\ e^v \left\{e^v - 1\right\}^{w-1} \int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-z}}\right\}^{i-w-1+w-1+2} \left\{e^{-z}\right\}^{w-1+n-i+2} \left\{\frac{1}{e^v+e^{-z}}\right\}^{w-1+n-i+2} dz \\
&= \frac{S_1}{\sigma}\ e^v \left\{e^v - 1\right\}^{w-1} \int_{-\infty}^{\infty} \left\{\frac{1}{1+e^{-z}}\right\}^{i} \left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w+1} \left\{e^{-z}\right\}^{n-i+w+1} dz
\end{aligned}$$

This has the form [1, (3.315.1)]

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{e^{-\mu x}}{\left(e^{\beta}+e^{-x}\right)^{\nu}\left(e^{\gamma}+e^{-x}\right)^{\rho}} dx &= e^{\gamma(\mu-\rho)-\beta\nu} B(\mu, \nu+\rho-\mu)\ {}_2F_1\left(\nu, \mu; \nu+\rho; 1-e^{\gamma-\beta}\right) \\
&= e^{\gamma(\mu-\rho)-\beta\nu}\ \frac{(\mu-1)!(\nu+\rho-\mu-1)!}{(\nu-\rho-1)!}\ {}_2F_1\left(\nu, \mu; \nu+\rho; 1-e^{\gamma-\beta}\right)
\end{aligned}$$

Letting $\beta = 0$, $\nu = i$, $\gamma = v$, $\rho = n-i+w+1$, and $\mu = n-i+w+1$, this satisfies the requirements for the definite integral. Further, $\nu+\rho-\mu-1 = i-1$, $\nu+\rho-1 = n+w$, and $\gamma(\mu-\rho)-\beta\nu = 0$, giving

$$\begin{aligned}
f_{D_{i,w,logis}}(y) &= \frac{S_1}{\sigma} e^v \left\{e^v-1\right\}^{w-1} B(n-i+w+1, i)\ {}_2F_1\left(i, n-i+w+1; n+w+1; 1-e^v\right) \\
&= \frac{n!}{(i-w-1)!(w-1)!(n-i)!} \frac{1}{\sigma} e^v \left\{e^v-1\right\}^{w-1} \frac{(n-i+w)!(i-1)!}{(n+w)!}\ {}_2F_1\left(i, n-i+w+1; n+w+1; 1-e^v\right) \\
&= \frac{n!}{(n+w)!} \frac{(i-1)!}{(i-w-1)!} \frac{(n-i+w)!}{(n-i)!} \frac{1}{(w-1)!} \frac{1}{\sigma} e^v \left\{e^v-1\right\}^{w-1} {}_2F_1\left(i, n-i+w+1; n+w+1; 1-e^v\right) \\
&= w \left\{\prod_{j=1}^{w} \frac{(i-j)(n-i+j)}{j(n+j)}\right\} \frac{1}{\sigma} e^v \left\{e^v-1\right\}^{w-1} {}_2F_1\left(i, n-i+w+1; n+w+1; 1-e^v\right) \qquad \text{(L.3)}
\end{aligned}$$

Expected Interval Spacing

Rather than attempting to integrate the hypergeometric function, we swap the order of the variables to get the expected interval spacing for the logistic function.

$$\begin{aligned}
E\left\{D_{i,w,logis}\right\} &= \int_{0}^{\infty} y\ dy\ f_{D_{i,w,logis}}(y) \\
&= \int_{0}^{\infty} \sigma^2 v\ dv \int_{-\infty}^{\infty} \frac{S_1}{\sigma} e^v \left\{e^v-1\right\}^{w-1} \left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w+1} \left\{e^{-z}\right\}^{n-i+w+1} dz \\
&= S_1 \sigma \int_{-\infty}^{\infty} dz \left\{\frac{1}{1+e^{-z}}\right\}^{i} \left\{e^{-z}\right\}^{n-i+w+1} \int_{0}^{\infty} v e^v \left\{e^v-1\right\}^{w-1} \left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w+1} dv \\
&= S_1 \sigma \int_{-\infty}^{\infty} dz \left\{\frac{1}{1+e^{-z}}\right\}^{i} \left\{e^{-z}\right\}^{n-i+w+1} I_{in}
\end{aligned}$$

To evaluate the inner integral, we will first evaluate by parts which will give a series, and then will expand a polynomial in a second series. First we will need a series solution to an indefinite integral. From [1, (2.153)]

$$\int \frac{(c+xd)^{\nu}}{(a+xb)^{\mu}}\ dx = \frac{1}{-(\mu-\nu-1)}\frac{1}{b}\frac{(c+dx)^{\nu}}{(a+xb)^{\mu-1}} - \frac{1}{-(\mu-\nu-1)}\frac{ad-bc}{b}\int \frac{(c+xd)^{\nu-1}}{(a+xb)^{\mu}}\ dx$$

In general, one step is

$$\begin{aligned}\int \frac{(c+xd)^{\nu-k}}{(a+xb)^{\mu}}\ dx &= \frac{1}{-(\mu-\nu+k-1)}\frac{1}{b}\frac{(c+xd)^{\nu-k}}{(a+xb)^{\mu-1}} - \frac{1}{-(\mu-\nu+k-1)}\frac{ad-bc}{b}\int \frac{(c+xd)^{\nu-(k+1)}}{(a+xb)^{\mu}}\ dx \\ &= -A_k + B_k\int \frac{(c+xd)^{\nu-(k+1)}}{(a+xb)^{\mu}}\ dx\end{aligned}$$

The recursion down the chain follows

$$\begin{aligned}\int \frac{(c+xd)^{\nu-k}}{(a+xb)^{\mu}} dx &= -A_0 + B_0\int \frac{(c+xd)^{\nu-1}}{(a+xb)^{\mu}}\ dx \\ &= -A_0 + B_0\left[-A_1 + B_1\int \frac{(c+xd)^{\nu-2}}{(a+xb)^{\mu}}\ dx\right] \\ &= -A_0 - A_1B_0 + B_0B_1\left[-A_2 + B_2\int \frac{(c+xd)^{\nu-3}}{(a+xb)^{\mu}}\ dx\right] \\ &= -A_0 - A_1B_0 - A_2B_0B_1 + B_0B_1B_2\left[-A_3 + B_3\int \frac{(c+xd)^{\nu-4}}{(a+xb)^{\mu}}\ dx\right]\end{aligned}$$

so at the last step, with $k=\nu$,

$$\int \frac{(c+xd)^{\nu-k}}{(a+xb)^{\mu}}\ dx = -A_0 - \sum_{k=1}^{\nu-1} A_k \prod_{l=0}^{k-1} B_l + \left\{\prod_{l=0}^{\nu-1} B_l\right\}\int \frac{dx}{(a+xb)^{\mu}}$$

This termination is, via [1, (2.111.1)]

$$\int (a+xb)^{\rho}\ dx = \frac{1}{b(\rho+1)}\ (a+xb)^{\rho+1}$$

with $\rho=-\mu$. This gives

$$\begin{aligned}\int \frac{dx}{(a+xb)^{\mu}} &= \frac{1}{b(-\mu+1)}\ (a+xb)^{-\mu+1} \\ &= \frac{1}{b(-\mu+1)}\frac{1}{(a+bx)^{\mu-1}} \\ &= -A_{\nu}\end{aligned}$$

The recursion chain is now

$$\begin{aligned}\int \frac{(c+xd)^{\nu}}{(a+xb)^{\mu}} dx &= -A_0 - \sum_{k=1}^{\nu-1} A_k \prod_{l=0}^{k-1} B_l - A_{\nu}\prod_{l=0}^{\nu-1} B_l \\ &= -A_0 - \sum_{k=1}^{\nu} A_k \prod_{l=0}^{k-1} B_l\end{aligned}$$

The product simplifies

$$\begin{aligned}\prod_{l=0}^{k-1} B_l &= \prod_{l=0}^{k-1} \frac{\nu-l}{\mu-\nu+l-1}\frac{ad-bc}{b} \\ &= \left(\frac{ad-bc}{b}\right)^{k}\frac{\nu(\nu-1)(\nu-2)\dots(\nu-k+1)}{(\mu-\nu-1)(\mu-\nu)(\mu-\nu+1)\dots(\mu-\nu+k-2)} \\ &= \left(\frac{ad-bc}{b}\right)^{k}\frac{\nu!}{(\nu-k)!}\frac{\mu-\nu-2)!}{(\mu-\nu+k-2)!}\end{aligned}$$

Substituting this back in,

$$\begin{aligned}
\int \frac{(c+xd)^\nu}{(a+xb)^\mu}\,dx &= -A_0 - \sum_{k=1}^{\nu} A_k \left(\frac{ad-bc}{b}\right)^k \frac{\nu!}{(\nu-k)!}\frac{\mu-\nu-2)!}{(\mu-\nu+k-2)!} \\
&= -A_0 - \sum_{k=1}^{\nu} \frac{1}{\mu-\nu+k-1}\frac{1}{b}\frac{(c+xd)^{\nu-k}}{(a+xb)^{\mu-1}} \left(\frac{ad-bc}{b}\right)^k \frac{\nu!}{(\nu-k)!}\frac{\mu-\nu-2)!}{(\mu-\nu+k-2)!} \\
&= -A_0 - \frac{\nu!(\mu-\nu-2)!}{b(a+xb)^{\mu-1}} \sum_{k=1}^{\nu} (c+xd)^{\nu-k} \left(\frac{ad-bc}{b}\right)^k \frac{1}{(\nu-k)!}\frac{1}{(\mu-\nu+k-1)!}
\end{aligned}$$

At $k=0$ the second term simplies to A_0 and we can fold it into the series. The final, general solution of this indefinite integral is

$$\int \frac{(c+xd)^\nu}{(a+xb)^\mu}\,dx = -\frac{\nu!(\mu-\nu-2)!}{b(a+bx)^{\mu-1}} \sum_{k=0}^{\nu} (c+xd)^{\nu-k} \left(\frac{ad-bc}{b}\right)^k \frac{1}{(\nu-k)!}\frac{1}{(\mu-\nu+k-1)!} \tag{L.4}$$

To solve I_{in} we work by parts, which will require (L.4). Let $t=v$, $dt=dv$, and

$$\begin{aligned}
du &= e^v\left\{e^v-1\right\}^{w-1}\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w-1} \\
&= \left\{r-1\right\}^{w-1}\left\{\frac{1}{r+e^{-z}}\right\}^{n-i+w+1}
\end{aligned}$$

making the further substitution $r=e^v$ and $dr=e^v dv$. This is in the form of (L.4) with $a=e^{-z}$, $b=1$, $c=-1$, $d=1$, $\nu=w-1$, $\mu=n-i+w+1$, and $\nu-\mu-1=n-i+1$. So

$$\begin{aligned}
u &= -(w-1)!(n-i)!\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w}\sum_{k=0}^{w-1}\frac{1}{(w-1-k)!}\frac{1}{(n-i+1+k)!}\left\{\frac{e^{-z}+1}{1}\right\}^k\left\{e^v-1\right\}^{w-1-k} \\
&= -(w-1)!(n-i)!\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w}\sum_{k=0}^{w-1}\frac{1}{(w-1-k)!}\frac{1}{(n-i+1+k)!}\left\{1+e^{-z}\right\}^k\left\{e^v-1\right\}^{w-1-k}
\end{aligned}$$

Then

$$\begin{aligned}
I_{in} &= \int_0^\infty v\ e^v\left\{e^v-1\right\}^{w-1}\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w+1} dv \\
&= [tu]_0^\infty - \int_0^\infty u\ dt \\
&= \left[-(w-1)!(n-i)!\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w} v\sum_{k=0}^{w-1}\frac{1}{(w-1-k)!}\frac{1}{(n-i+1+k)!}\left\{1+e^{-z}\right\}^k\left\{e^v-1\right\}^{w-1-k}\right]_0^\infty \\
&\quad + \int_0^\infty (w-1)!(n-i)!\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w}\sum_{k=0}^{w-1}\frac{1}{(w-1-k)!}\frac{1}{(n-i+1+k)!}\left\{1+e^{-z}\right\}^k\left\{e^v-1\right\}^{w-1-k} \\
&= \int_0^\infty (w-1)!(n-i)!\sum_{k=0}^{w-1}\frac{1}{(w-1-k)!}\frac{1}{(n-i+1+k)!}\left\{1+e^{-z}\right\}^k\left\{e^v-1\right\}^{w-1-k}\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w} dv \\
&= \sum_{k=0}^{w-1}\frac{(w-1)!}{(w-1-k)!}\frac{(n-i)!}{(n-i+1+k)!}\left\{1+e^{-z}\right\}^k\int_0^\infty\left\{e^v-1\right\}^{w-1-k}\left\{\frac{1}{e^v+e^{-z}}\right\}^{n-i+w} dv
\end{aligned}$$

The square brackets in the third step goes to zero at both bounds, with $1/e^v$ dominating at infinity and v at 0.

To evaluate this integral we now expand the first factor as a polynomial using [1, (1.111)]

$$\begin{aligned}
(a+x)^\nu &= \sum_{l=0}^{\nu}\binom{\nu}{l}x^l a^{\nu-l} \\
&= a^\nu + \nu a^{\nu-1}x + \sum_{l=2}^{\nu}\binom{\nu}{l}a^{\nu-l}x^l
\end{aligned}$$

The first two terms have been expanded from the series because they will integrate simply. We have $\nu = w - 1 - k$ and $a = -1$ for $x = e^v$. The second term will disappear if $\nu = 0$ or $k = w - 1$. Ignore the sum if $\nu < 2$, which occurs at $w = 2$ or $k > w - 2$. So

$$\begin{aligned}
&\int_0^\infty \{e^v - 1\}^{w-1-k} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
&\quad = \int_0^\infty (-1)^{w-1-k} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
&\qquad + \int_0^\infty (w-1-k)(-1)^{w-2-k} e^v \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
&\qquad + \sum_{l=2}^{w-1-k} \binom{w-1-k}{l} (-1)^{w-1-k} \int_0^\infty \{e^v\}^l \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv
\end{aligned}$$

For the first integral we substitute $r = e^v + e^{-z}$, $dr = e^v dv = (r - e^{-z}) dv$. We will also need [1, (2.117.4)]

$$\begin{aligned}
\int \frac{dx}{x^\nu (a + xb)} &= \frac{(-1)^\nu b^{\nu-1}}{a^\nu} \ln \frac{a + xb}{x} + \sum_{l=1}^{\nu-1} \frac{(-1)^l b^{l-1}}{(\nu - l)} a^l \frac{1}{x^{\nu-l}} \\
&= \frac{1}{b} \left(-\frac{b}{a}\right)^\nu \ln \frac{a + xb}{x} + \sum_{l=1}^{\nu-1} \frac{1}{\nu - l} \frac{1}{b} \left(-\frac{b}{a}\right)^l \left(\frac{1}{x}\right)^{\nu-l}
\end{aligned}$$

letting $\nu = n - i + w$, $a = -e^{-z}$, and $b = 1$.

$$\begin{aligned}
&\int_0^\infty \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
&\quad = \int \left\{\frac{1}{r}\right\}^{n-i+w} \left\{\frac{1}{r - e^{-z}}\right\} dr \\
&\quad = \left\{\frac{1}{e^{-z}}\right\}^{n-i+w} \ln \frac{r - e^{-z}}{r} + \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-l} \left\{\frac{1}{e^{-z}}\right\}^l \left\{\frac{1}{r}\right\}^{n-i+w-l} \\
&\quad = \left[\left\{\frac{1}{e^{-z}}\right\}^{n-i+w} \ln \frac{e^v}{e^v + e^{-z}} + \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-l} \left\{\frac{1}{e^{-z}}\right\}^l \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w-l}\right]_0^\infty \\
&\quad = \left\{\frac{1}{e^{-z}}\right\}^{n-i+w} \ln\left\{1 + e^{-z}\right\} + \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-l} \left\{\frac{1}{e^{-z}}\right\}^l \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-l}
\end{aligned}$$

At $v = \infty$ both terms go to 0, and at the lower bound e^v is 1.

For the second integral we substitute $r = e^v$, $dr = e^v dv$, and use [1, (2.111.1)]

$$\int (a + xb)^\mu dx = \frac{1}{b(\mu + 1)} (a + xb)^{\mu+1}$$

letting $\mu = -(n - i + w)$, $a = e^{-z}$, and $b = 1$.

$$\begin{aligned}
\int_0^\infty e^v \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv &= \int \left\{\frac{1}{r + e^{-z}}\right\}^{n-i+w} dr \\
&= \left[\frac{1}{-(n-i+w+1)} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w-1}\right]_0^\infty \\
&= \frac{1}{n-i+w-1} \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-1}
\end{aligned}$$

where the integral is zero at the upper bound and again $e^v \to 1$ at the lower.

After making the same r substitution, the third integral takes the form of (L.4) with $a = e^{-z}$, $b = 1$, $c = 0$, $d = 1$, $\mu = n - i + w$, $\nu = l - 1$, and $\mu - \nu - 1 = n - i + w - l$.

$$\begin{aligned}
&\int_0^\infty \{e^v\}^{l-1} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} e^v \, dv \\
&\quad = \int r^{l-1} \left\{\frac{1}{r + e^{-z}}\right\}^{n-i+w} dr \\
&\quad = \left[-(l-1)!(n-i+w-l-1)! \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w-1} \sum_{j=0}^{l-1} \frac{1}{(l-1-j)!} \frac{1}{(n-i+w-l+j)!} \left\{e^{-z}\right\}^j \left\{e^v\right\}^{l-1-j} \right]_0^\infty \\
&\quad = (l-1)!(n-i+w2-l-1)! \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-1} \sum_{j=0}^{l-1} \frac{1}{(l-1-j)!} \frac{1}{(n-i+w-l+j)!} \left\{e^{-z}\right\}^j
\end{aligned}$$

with the same behavior at the upper and lower bounds.

Assembling the three integrals, we get

$$\begin{aligned}
I_{in} &= \int_0^\infty v e^v \left\{e^v - 1\right\}^{w-1} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w+1} dv \\
&= \sum_{k=0}^{w-1} S_3 \left\{1 + e^{-z}\right\}^k \int_0^\infty \left\{e^v - 1\right\}^{w-1-k} \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
&= \sum_{k=0}^{w-1} S_3 \left\{1 + e^{-z}\right\}^k \left[\begin{array}{r}
(-1)^{w-1-k} \int_0^\infty \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
+(w-1-k)(-1)^{w-2-k} \int_0^\infty e^v \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv \\
+ \sum_{l=2}^{w-1-k} \binom{w-1-k}{l} (-1)^{w-1-k-l} \int_0^\infty \left\{e^v\right\}^l \left\{\frac{1}{e^v + e^{-z}}\right\}^{n-i+w} dv
\end{array} \right] \\
&= \sum_{k=0}^{w-1} S_3 (-1)^{w-1-k} \left\{1 + e^{-z}\right\}^k \left[\begin{array}{r}
\left\{\frac{1}{e^{-z}}\right\}^{n-i+w} \ln\left\{1 + e^{-z}\right\} \\
- \left\{ \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-1} \left\{\frac{1}{e^{-z}}\right\}^l \left\{\frac{1}{1 + e^{-z}}^{n-i+w-l}\right\} \right\} \\
- \frac{w-1-k}{n-i+w-1} \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-1} \\
+ \left\{ \begin{array}{r}
\sum_{l=2}^{w-1-k} \binom{w-1-k}{l} (-1)^l (l-1)!(n-i+w-l-1)! \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-1} \\
\cdot \sum_{j=0}^{l-1} \frac{1}{(l-1-j)!} \frac{1}{(n-i+w-l-j)!} \left\{e^{-z}\right\}^j
\end{array} \right\}
\end{array} \right] \\
&= \sum_{k=0}^{w-1} S_3 (-1)^{w-1-k} \left[\begin{array}{r}
\left\{1 + e^{-z}\right\}^k \left\{\frac{1}{e^{-z}}\right\}^{n-i+w} \ln\left\{1 + e^{-z}\right\} \\
- \left\{ \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-l} \left\{\frac{1}{e^{-z}}\right\}^l \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-k-l} \right\} \\
- \frac{w-1-k}{n-i+w-1} \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-k-1} \\
+ \sum_{l=2}^{w-1-k} \binom{w-1-k}{l} (-1)^l \sum_{j=0}^{l-1} \frac{(l-1)!}{(l-1-j)!} \frac{(n-i+w-l-1)!}{(n-i+w-l+j)!} \left\{\frac{1}{1 + e^{-z}}\right\}^{n-i+w-1-k} \left\{e^{-z}\right\}^l
\end{array} \right]
\end{aligned} \tag{L.5}$$

$$S_3 = \frac{(w-1)!}{(w-1-k)!} \frac{(n-i)!}{(n-i+1+k)!}$$

We can simplify the scaling factors in the last term, canceling the $(l-1)!$ from the binomial coefficient and leaving a factor of $1/l$. For calculations the fractions of similar factorials reduce further to products.

$$\begin{aligned}
&\binom{w-1-k}{l}\frac{(l-1)!}{(l-1-j)!}\frac{(n-i+w-1-l)!}{(n-i+w-l+j)!}\\
&\qquad=\frac{(w-1-k)!}{l!(w-1-k-l)!}\frac{(l-1)!}{(l-1-j)!}\frac{(n-i+w-l-1)!}{(n-i+w-l+j)!}\\
&\qquad=\frac{1}{l}\frac{1}{(l-1-j)!}\frac{(w-1-k)!}{(w-1-k-l)!}\frac{(n-i+w-l-1)!}{(n-i+w-l+j)!}\\
&\qquad=\frac{1}{l}\frac{1}{(l-1-j)!}\left[\prod_{p=0}^{l-1} w-1-k-p\right]\left[\prod_{p=0}^{j}\frac{1}{n-i+w-j+p}\right]
\end{aligned}$$

The expected interval spacing is now, with the inner integral known,

$$\begin{aligned}
E\Big\{D_{i,w,logis}\Big\} &= S_1\sigma\int_{-\infty}^{\infty} dz\left\{\frac{1}{1+e^{-z}}\right\}^i\left\{e^{-z}\right\}^{n-i+w+1} I_{in}\\
&= S_1\sigma\int_{-\infty}^{\infty} dz\left\{\frac{1}{1+e^{-z}}\right\}^i\left\{e^{-z}\right\}^{n-i+w+1}\sum_{k=0}^{w-1}\frac{(w-1)!}{(w-1-k)!}\frac{(n-i)!}{(n-i+1+k)!}\\
&\quad\cdot\Bigg[\left\{\frac{1}{1+e^{-z}}\right\}^{-k}\left\{e^{-z}\right\}^{-n+i-w}\ln\left\{1+e^{-z}\right\}\\
&\qquad-\left\{\sum_{l=1}^{n-i+w-1}\frac{1}{n-i+w-l}\left\{\frac{1}{1+e^{-z}}\right\}^{n-i+w-k-l}\left\{e^{-z}\right\}^{-l}\right\}\\
&\qquad-\frac{w-1-k}{n-i+w-1}\left\{\frac{1}{1+e^{-z}}\right\}^{n-i+w-k-1}\\
&\qquad+\sum_{l=2}^{w-1-k}(-1)^l\sum_{j=0}^{l-1}\frac{1}{l}\frac{1}{(l-1-j)!}\frac{(w-1-k)!}{(w-1-k-l)!}\frac{(n-i+w-l-1)!}{(-i+w-l+j)!}\left\{\frac{1}{1+e^{-z}}\right\}^{n-i+w-1-k}\left\{e^{-z}\right\}^{-j}\Bigg]\\
&=\sum_{k=0}^{w-1}\frac{n!}{(i-w-1)!(w-1-k)!(n-i+1+k)!}\sigma(-1)^{w-1-k}\\
&\quad\cdot\Bigg[\int_{-\infty}^{\infty}dz\left\{\frac{1}{1+e^{-z}}\right\}^{i-k}\left\{e^{-z}\right\}^{i}\ln\left\{1+e^{-z}\right\}\\
&\qquad-\sum_{l=1}^{n-i+w-1}\frac{1}{n-i+w-l}\int_{-\infty}^{\infty}dz\left\{\frac{1}{1+e^{-z}}\right\}^{n+w-k-l}\left\{e^{-z}\right\}^{n-i+w+1-l}\\
&\qquad-\frac{w-1-k}{n-i+w-1}\int_{-\infty}^{\infty}dz\left\{\frac{1}{1+e^{-z}}\right\}^{n+w-k-1}\left\{e^{-z}\right\}^{n-i+w+1}\\
&\qquad+\sum_{l=2}^{w-1-k}(-1)^l\sum_{j=0}^{l-1}\frac{1}{l}\frac{1}{(l-1-j)!}\frac{(w-1-k)!}{(w-1-k-l)!}\frac{(n-i+w-l-1)!}{(n-i+w-l+j)!}\\
&\qquad\cdot\int_{-\infty}^{\infty}dz\left\{\frac{1}{1+e^{-z}}\right\}^{n+w-1-k}\left\{e^{-z}\right\}^{n-i+w+1+j}\Bigg]
\end{aligned}$$

There are four integrals in the last lines, but only two unique forms. For the first term we will use [1, (4.293.14)]

$$\int_0^{\infty}\frac{x^{\nu-1}}{(\gamma+x)^{\mu}}\ \ln(\gamma+x)\ dx=\gamma^{\nu-\mu}B(\nu,\mu-\nu)\left[\psi(\mu)-\psi(\mu-\nu)+\ln\gamma\right]$$

along with $\nu=1$, $\mu=i-k$, $\mu-\nu=i-k-1$. The conditions on the integral, $\mu>\nu>0$ hold at all k, since $i>w$. The first term becomes

$$\begin{aligned}
\int_{-\infty}^{\infty} dz \left\{\frac{1}{1+e^{-z}}\right\}^{i-k} e^{-z} \ln\left\{1+e^{-z}\right\} &= 1^{k+1-i} B(1, i-k-1)\left[\psi(i-k) - \psi(i-k-1) + \ln 1\right] \\
&= B(1, i-k-1)\left[\psi(i-k) - \psi(i-k-1)\right] \\
&= \frac{1}{i-k-1}\left[\psi(i-k) - \psi(i-k-1)\right]
\end{aligned}$$

The other three integrals fit [1, (8.380.3)]

$$\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\, dx = B(x,y) = \frac{(x-1)!(y-1)!}{(x+y-1)!}$$

This would also follow from (L.4), because at the upper bound the terms are zero and at the lower bound only the lead, non-series term survives. This is equivalent to the beta function. We make the substitution $r = e^{-z}$, $dr = -e^{-z}dz$ to transform

$$\begin{aligned}
\int_{-\infty}^{\infty} dz \left\{\frac{1}{1+e^{-z}}\right\}^{\mu} \left\{e^{-z}\right\}^{\nu} &= \int_{\infty}^{0} -dr \left\{\frac{1}{1+r}\right\}^{\mu} r^{\nu-1} \\
&= \int_{0}^{\infty} dr \left\{\frac{1}{1+r}\right\}^{\mu} r^{\nu-1} \\
&= B(\nu, \mu-\nu) \\
&= \frac{(\nu-1)!(\mu-\nu-1)!}{(\mu-1)!}
\end{aligned}$$

having let $x = \nu$ and $y = \mu - \nu$ for the known integral. We have for the

second term	$\nu = n-i+w+1-l$	$\mu = n+w-k-l$	$\mu-\nu = i-k-1$
third term	$\nu = n-i+w+1$	$\mu = n+w-1-k$	$\mu-\nu = i-k-2$
fourth term	$\nu = n-i+w+1+j$	$\mu = n+w-1-k$	$\mu-\nu = i-k-2-j$

The final solution for the expected interval spacing is

$$\begin{aligned}
E\Big\{D_{i,w,logis}\Big\} = \sum_{k=0}^{w-1} & \frac{n!}{(i-w-1)!(w-1-k)!(n-i+1+k)!}\sigma(-1)^{w-1-k} \\
\cdot & \left[\begin{array}{r}
\frac{1}{i-k-1}\left[\psi(i-k) - \psi(i-k-1)\right] \\
- \sum_{l=1}^{n-i+w-1} \frac{1}{n-i+w-l}\, B(n-i+w+1-l, i-k-1) \\
- \frac{w-1-k}{n-i+w-1}\, B(n-i+w+1, i-k-2) \\
+ \sum_{l=2}^{w-1-k} (-1)^l \sum_{j=0}^{l-1} \frac{(w-1-k)!}{(w-1-k-l)!} \frac{(n-i+w-l-1)!}{(n-i+w-l+j)!} \frac{B(n-i+w+1+j, i-k-2-j)}{l(l-1-j)!}
\end{array}\right]
\end{aligned} \tag{L.6}$$

References

[1] Gradshteyn, I. S. and Ryzhik, I. M. (1980). *Table of Integrals, Series, and Products*, Corrected and Enlarged ed. Academic Press.

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